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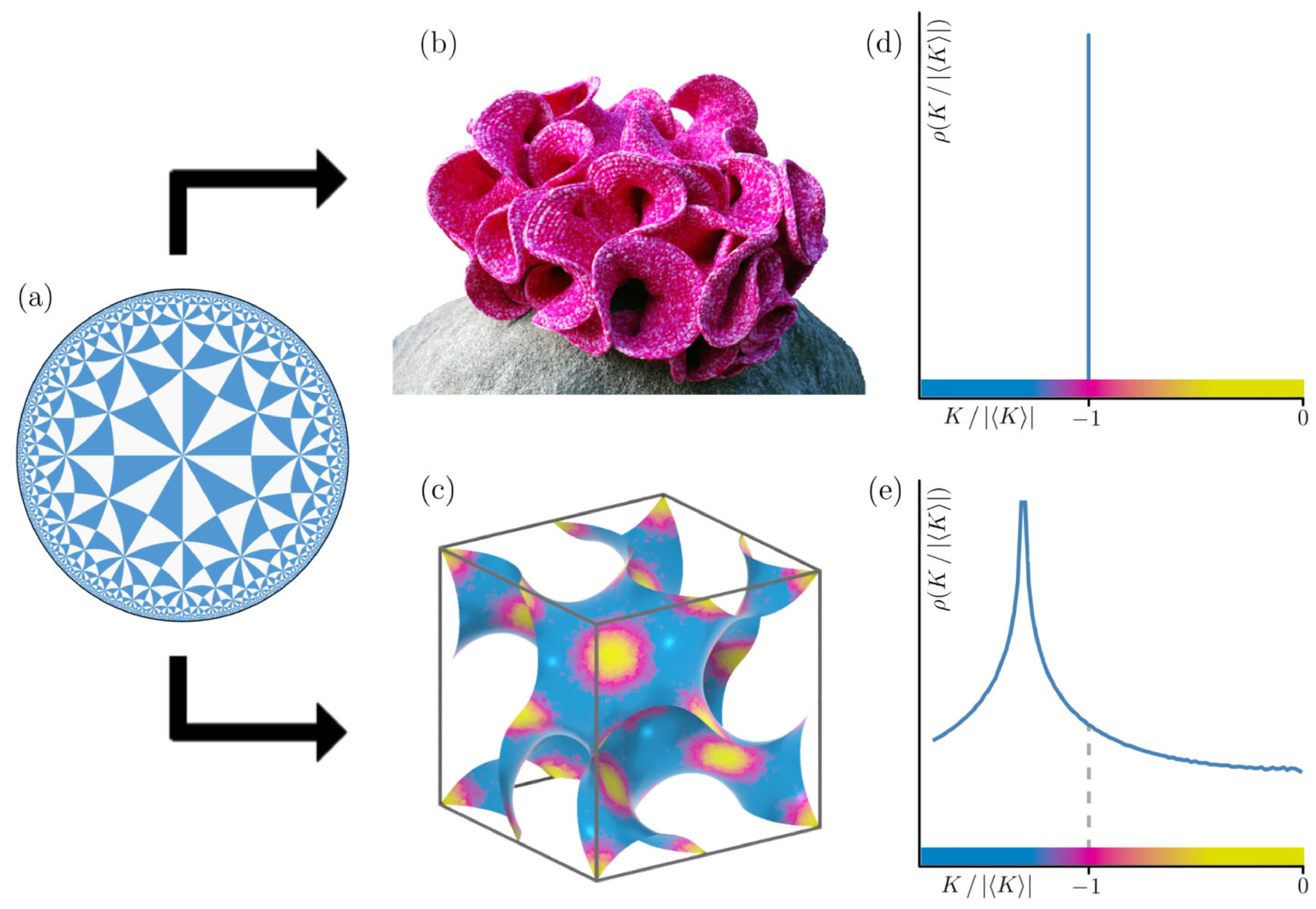
Exploring the Homogeneity of Disordered Minimal Surfaces†

1. Hilbert's Embedding Theorem



Hilbert's Embedding Theorem states that no complete regular isometric immersion of the hyperbolic plane $\mathbb{H}^2$ (a) into $\mathbb{R}^3$ exists. Daina Taimiņa's crocheted pseudosphere[1] (b) and the Gyroid (c) weaken this theorem's assumption by containing geometric singularities or Gaussian curvature fluctuations.

2. Homogeneity Measures for Surfaces

For a surface $\mathcal{M}$, denote the average Gaussian curvature by $\langle K\rangle$ and the Gaussian curvature variance by $\sigma = \sqrt{\langle K^2\rangle - \langle K\rangle^2}$. Then, the *fluctuation of Gaussian curvature*[2] is given by

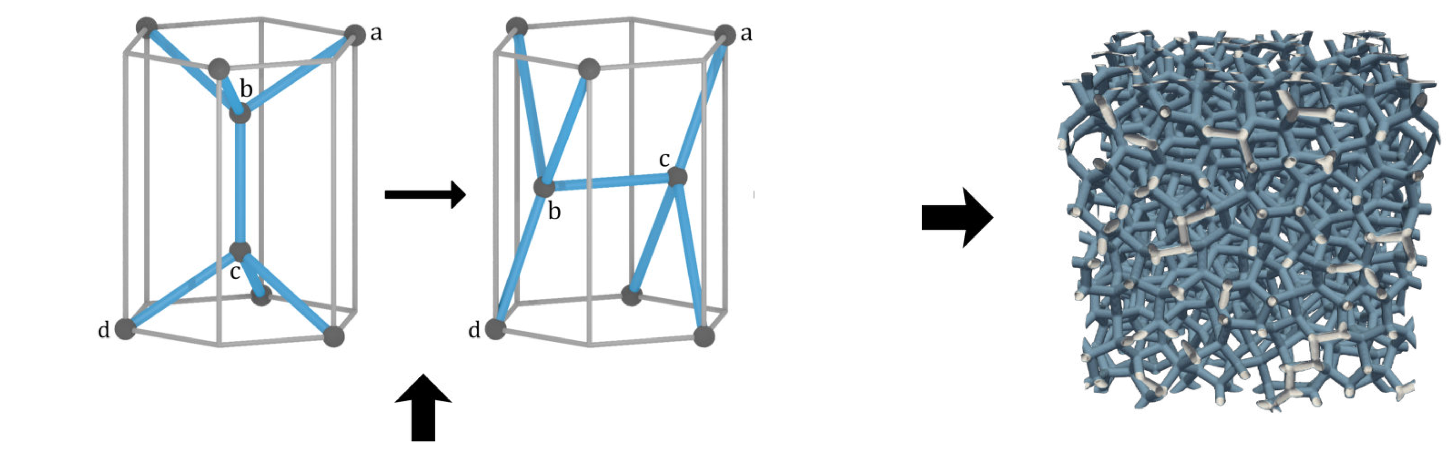
$$\frac{\sigma^2}{\langle K\rangle^2} = \frac{\langle K^2\rangle - \langle K\rangle^2}{\langle K\rangle^2} = A(\mathcal{M}) \cdot \frac{\int_{\mathcal{M}} K^2\, dA}{\left(\int_{\mathcal{M}} K\, dA\right)^2} - 1.$$

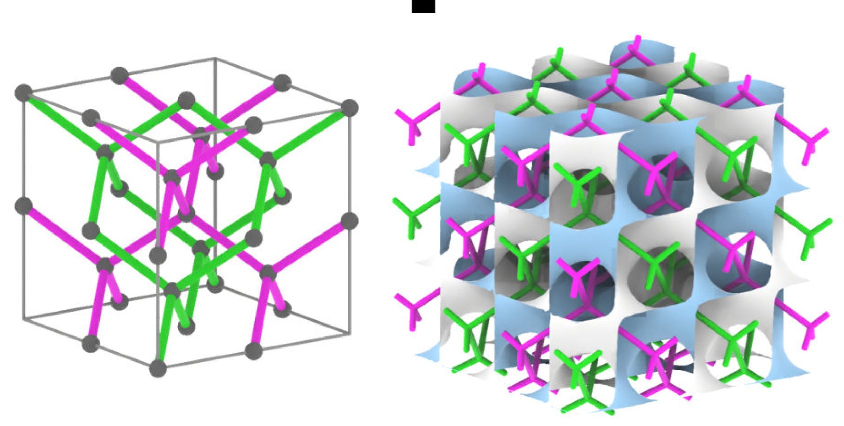
Conversely, we define the *isotropy index*[3] of the surface, measuring the variations of the unit normal field $\mathbf{n}$, as

$$\beta_1^{0,2}(\mathcal{M}) = \left|\frac{\lambda_3(W_1^{0,2}(\mathcal{M}))}{\lambda_1(W_1^{0,2}(\mathcal{M}))}\right| \quad \text{for} \quad W_1^{0,2}(\mathcal{M}) = \frac{1}{3}\int_{\mathcal{M}} \mathbf{n} \otimes \mathbf{n}\, dA$$

with the largest and smallest eigenvalues λ_1 and λ_3, respectively. Both quantities are dimensionless[4].

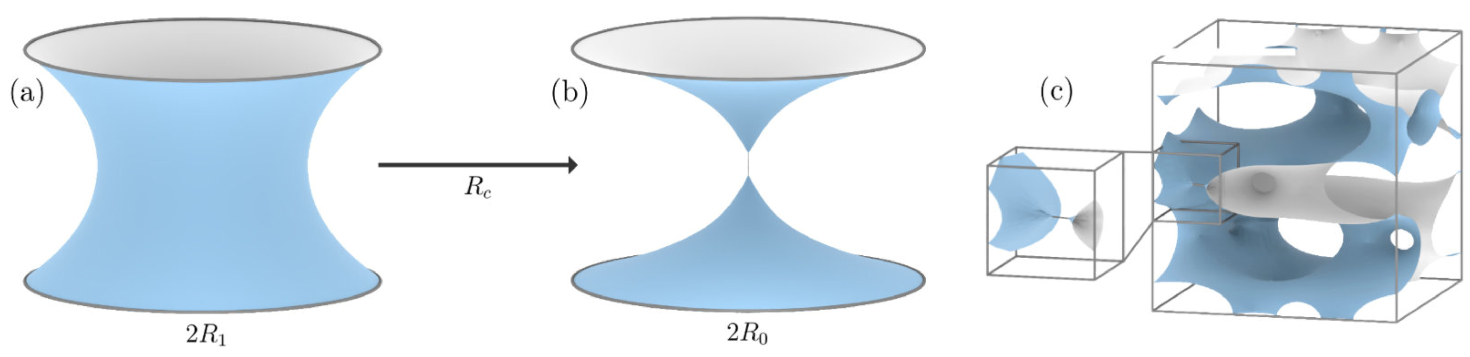
3. Amorphous Diamond Minimal Surface





By applying the Wooton-Winer-Weaire algorithm[5] to the cubic Diamond, we are able to generate an amorphous Diamond net. It can be tubified to create an initial configuration for curvature optimization algorithms of surface meshes.

4. Topology-Stabilized Curvature Optimization



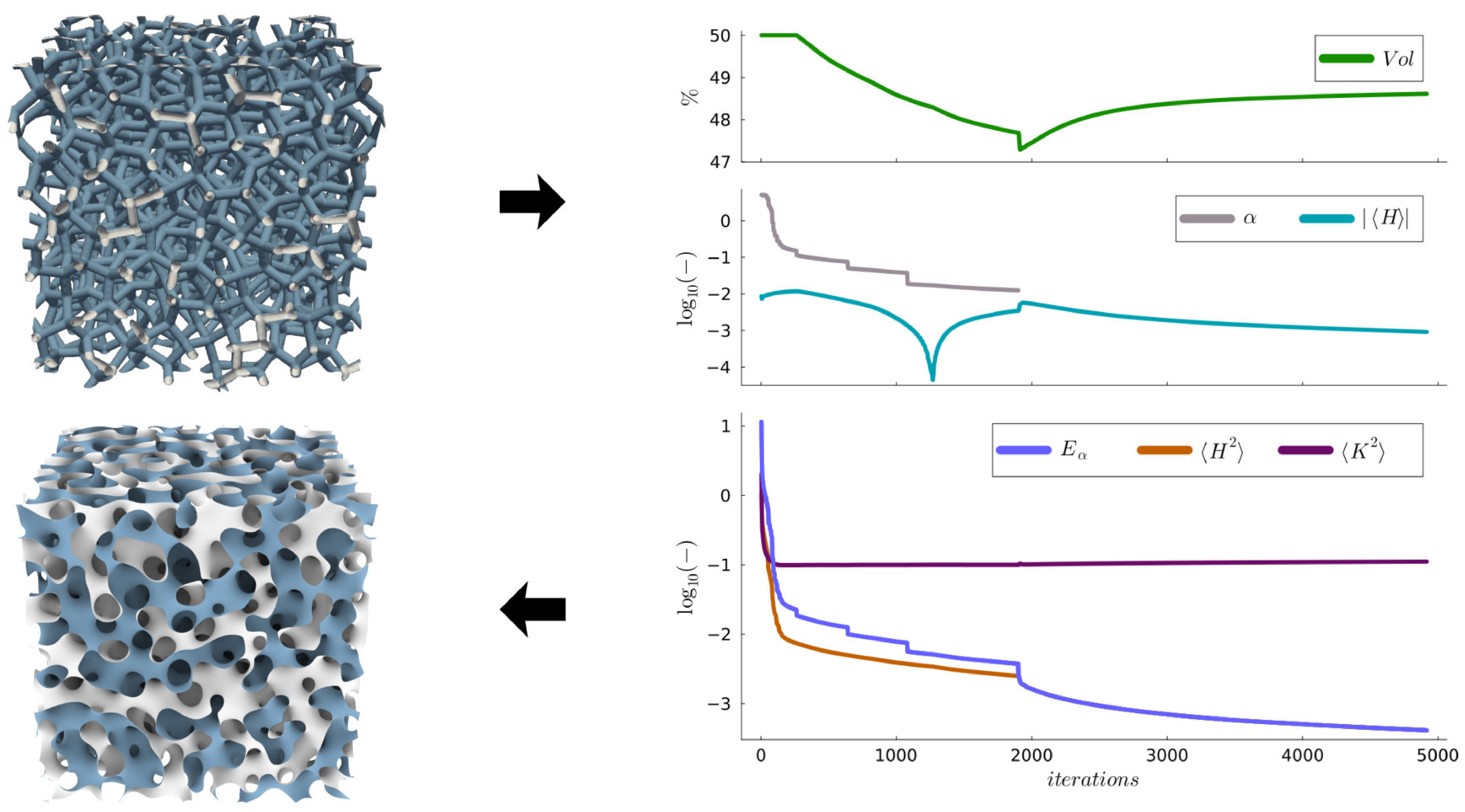
There exists a critical wireframe radius R_c below which the catenoid "pinches off" (a-b), wanting to attain the Goldschmidt solution. We observed this behavior in our experiments as well (c).

To prevent channel collapse and to avoid local minima, we propose a new objective function for use in the Surface Evolver[6]:

$$E_\alpha(\mathcal{M}) = \int_{\mathcal{M}} H^2\, dA + \alpha \cdot \int_{\mathcal{M}} K^2\, dA$$

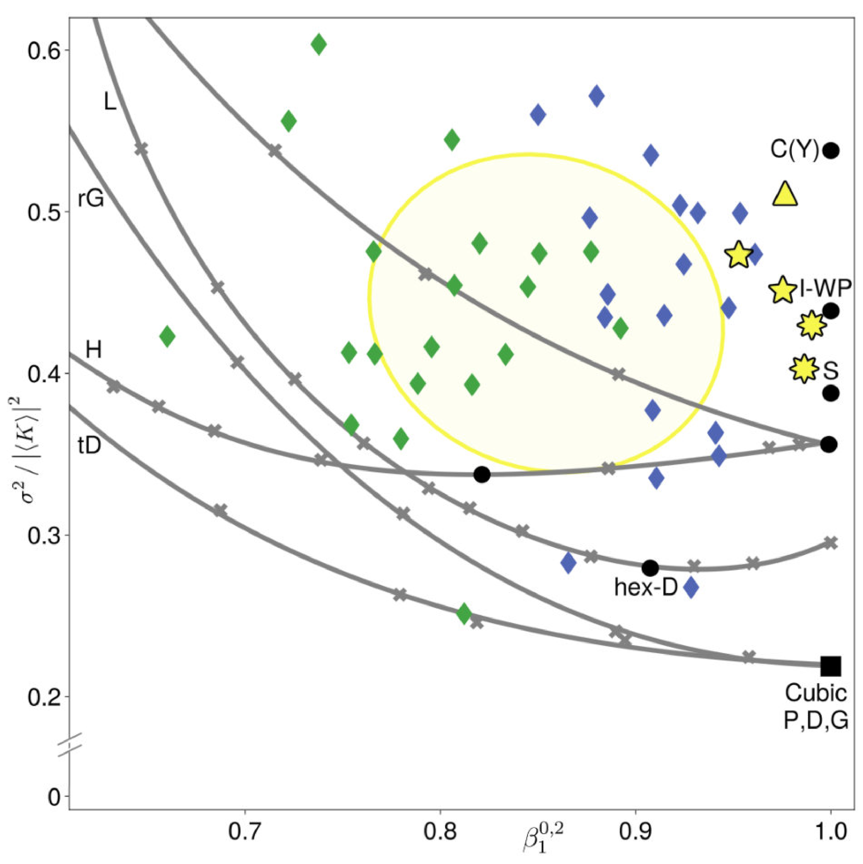
$$\text{with} \quad \alpha = \min\left\{\alpha',\ w \cdot \alpha' \cdot \frac{\int_{\mathcal{M}} H^2\, dA}{\int_{\mathcal{M}} K^2\, dA}\right\}$$

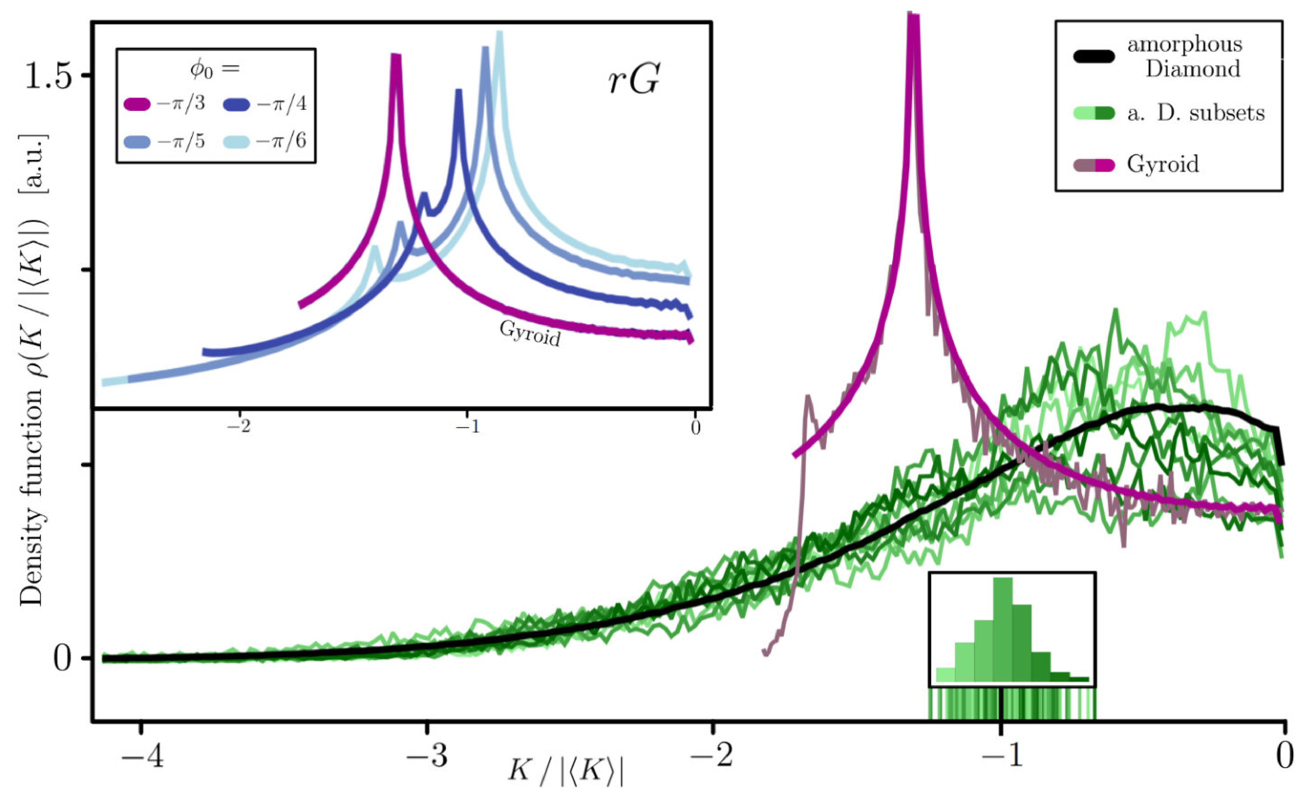
for the previous elastic modulus α' and damping factor $w > 0$.



5. Superior Uniformity of the Gyroid

We sample 5 amorphous Diamond surfaces in the sizes $\chi = -432$ (triangle), $\chi = -2000$ (5-star) and $\chi = -8192$ (8-star), normalizing them to $\langle K\rangle = -1$. From these surfaces, we cut out random cubical subsamples with Euler characteristic $\chi = -16$ (green) and $\chi = 2^3 \cdot (-16)$ (blue). None of the amorphous minimal surfaces come close in curvature uniformity to the cubic Gyroid. In fact, we find that the Gyroid is particularly good at associating a minimal amount of its surface area to flat points and high-curvature regions.





†This project is joint work with M.C. Pedersen, M.E. Evans, M.A. Klatt, Philipp Schönhöfer, G.E. Schröder-Turk.
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