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combinatorial synergies

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Positivity, Convexity and Computation in Rigidity Theory

1. Geometric Constraint Systems

Definition: A *d-dimensional bar-joint framework* $\mathcal{F} = (G, p)$ is given by a finite simple graph $G = (V, E)$ and a *realization* $p : V \to \mathbb{R}^d$ so that $|V| = n$. We consider the polynomial system $g = (g_{ij})_{ij \in E} : \mathbb{R}^{dn} \to \mathbb{R}^{|E|}$ defined by

$$g_{ij}(\mathbf{p}) = \sum_{k=1}^{d} (\mathbf{p}_{ik} - \mathbf{p}_{jk})^2 - \ell_{ij}^2 \quad \text{for } ij \in E,$$

edge lengths $\ell_{ij} = ||p(i) - p(j)||$ and variables $\mathbf{p} = (\mathbf{p}_{jk} : j \in [n], k \in [d])$ representing the realization p. We denote the *realization space* of $\mathcal{F}$ by $g^{-1}(0)$.

Definition: A *d-dimensional polytope* $\mathcal{P}$ is given by a vertex set V, a facets set F, a vertex-facet incidence relation $\sim$ and a *realization* (p, a), where $p : V \to \mathbb{R}^d$ and $a : F \to \mathbb{R}^d \setminus \{0\}$ (a describes the outer facet normals), ensuring that $\langle p_i - p_j, a_k \rangle = 0$ whenever $i, j \sim k$. An edge set E can be derived from the structure. The constraints g of $\mathcal{P}$ are chosen so that all edges are of a prescribed length and all facets F are flat.

Since many methods apply to both, polytopes and bar-joint frameworks, we refer to them as *d-dimensional geometric constraint systems* (GCSs).

2. Rigidity and Flexibility

Definition: Let $\mathcal{F}$ be a *d-dimensional* GCS with realization p, vertices V and constraints g. A *continuous motion* is a continuous $\alpha : [0, 1] \to \mathbb{R}^n$ such that

▶ $\alpha(0) = p$, $\alpha(t)$ is a realization of $\mathcal{F}$ and
▶ $\alpha(t)$ satisfies the polynomial constraints g for all $t \in [0, 1]$.

α is called *trivial* if p and $\alpha(t)$ have the same pairwise Euclidean distances between vertices for every $t \in [0, 1]$. It is called *nontrivial* otherwise. If a nontrivial continuous motion exists, $\mathcal{F}$ is called *flexible*; otherwise, it is called *rigid*.
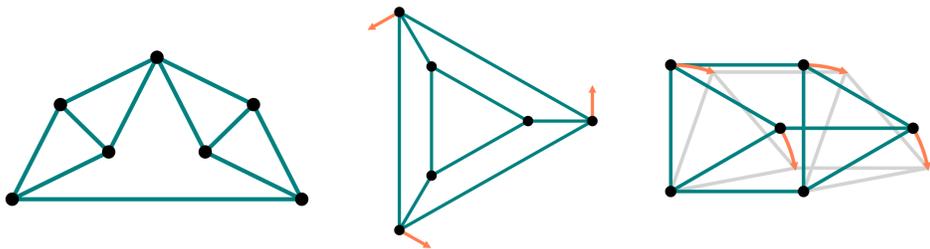

3. Infinitesimal Rigidity

Definition: Consider a GCS $\mathcal{F}$ in $\mathbb{R}^d$ with a polynomial constraint system $g = 0$ containing m equations and realization p mapping into $\mathbb{R}^{dn}$. The corresponding *rigidity matrix* is the Jacobian

$$R_{\mathcal{F}}(p) = \left(\frac{\partial g_i}{\partial \mathbf{p}_{jk}}(p) \right)_{i \in [m], (j,k) \in [n] \times [d]} \in \mathbb{R}^{m \times dn}.$$

An *infinitesimal flex* $\dot{p}$ is an element of $\ker R_{\mathcal{F}}(p)$ and an *equilibrium stress* ω is an element of $\operatorname{coker} R_{\mathcal{F}}(p)$. We call an infinitesimal flex *trivial* if it extends to an ambient isometry, and *nontrivial* otherwise. If each infinitesimal flex is trivial, we call the framework *infinitesimally rigid*; otherwise, it is *infinitesimally flexible*.

Theorem[1,2]: For a GCS $\mathcal{F}$, infinitesimal rigidity implies rigidity. This implication is generally not reversible.

Theorem: Every generic, convex realization of a 3-dimensional polytope is infinitesimally rigid[3]. Yet, flexible polytopes (both simplicial and convex) exist[3,4].

4. Second-Order Rigidity

Definition Consider a GCS $\mathcal{F} = (V, g, p)$. We say that an infinitesimal flex $\dot{p}$ is *blocked* by an equilibrium stress ω if

$$\omega^\top \cdot R_{\mathcal{F}}(\dot{p}) \cdot \dot{p} \neq 0.$$

$\mathcal{F}$ is called *second-order rigid* if every nontrivial infinitesimal flex is blocked by an equilibrium stress; it is called *prestress stable* if there is a single equilibrium stress that blocks all nontrivial infinitesimal flexes.

Theorem[5,6]: For GCS, second-order rigidity implies rigidity.
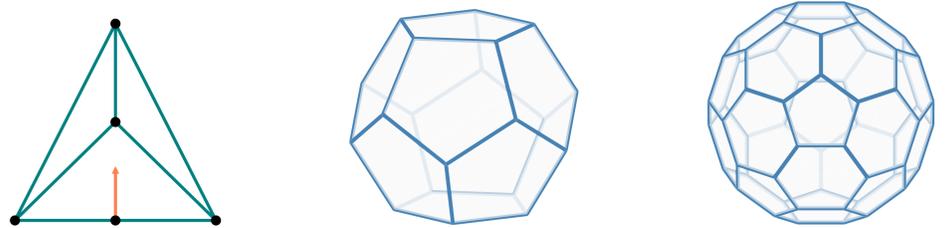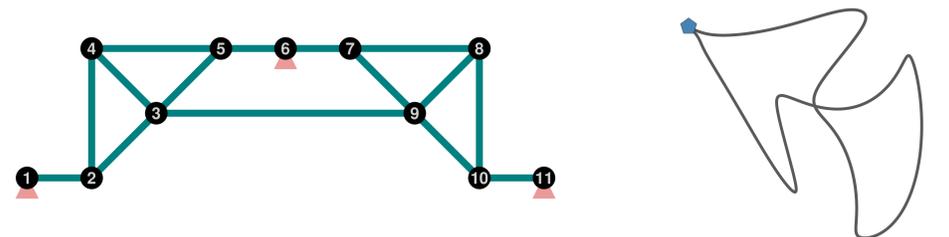

5. Higher-Order Rigidity

Definition: Given a GCS $\mathcal{F}$ with polynomial constraints g and realization p, we say that $(p^{(1)}, \ldots, p^{(N)}) \in \prod_{i=1}^{N} \mathbb{R}^{nd}$ is an *N-th order flex* if it satisfies

$$\sum_{k=0}^{n} \binom{n}{k} \cdot R_{\mathcal{F}}(p^{(k)}) \cdot p^{(n-k)} = 0$$

for $n = 1, \ldots, N$. $\mathcal{F}$ is called *N-order rigid* [7] if every N-th order flex $p^{(N)}$ comes with a trivial infinitesimal flex $p^{(1)}$.


6. Coned Polytopes

Definition: For a graph $G = (V, E)$ and $v \notin V$, the *cone graph* $G * \{v\}$ has the vertices $V \cup \{v\}$ and edges $E \cup \{iv : i \in V\}$. Given a bar-joint framework $\mathcal{F} = (G, p)$, the *cone framework* $\mathcal{F}^* = (G * \{v\}, p^*)$ is given by the realization p^* so that $p^*(u) = p(u)$ for $u \in V$ and $p^*(v)$ is the position of the *cone point*.

Theorem[8,9]: Coning the 1-skeleton of an affinely-spanning convex polytope from the inside produces a cone framework that is prestress stable.

Question: Take the 1-skeleton of a convex polytope. Where does placing the cone point result in a rigid framework? We call the surface of phase transitions between rigid and flexible realizations the *PSS Surface*.
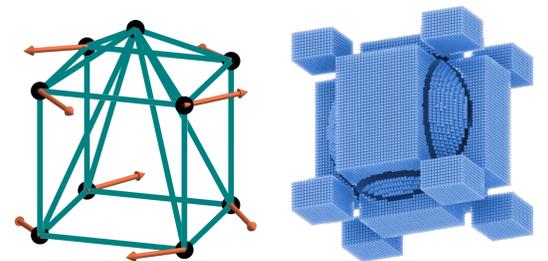
7. The Curious Case of the Truncated Dodecahedron
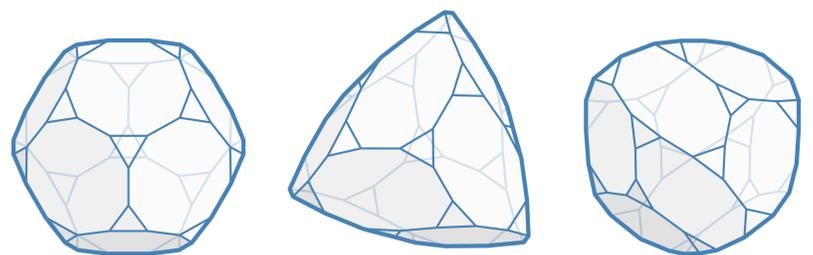
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