

Riemannian Optimization over Semialgebraic Sets[†]

1. Constrained Polynomial Optimization

Consider the constrained optimization problem

$$\min_{x \in \mathcal{C}} f(x) \quad (1)$$

for a polynomial $f \in \mathbb{R}[x_1, \dots, x_n]$ and a semialgebraic set \mathcal{C} generated by equalities $g_1(x), \dots, g_s(x) = 0$ and inequalities $h_1(x), \dots, h_t(x) \geq 0$ for polynomials $g_i, h_j \in \mathbb{R}[x_1, \dots, x_n]$.

A major obstacle in finding the optimum of a constrained polynomial optimization problem is that both the polynomial p and the constraint set \mathcal{C} are not convex in general. Solution strategies involve reformulating this optimization problem as the maximization of λ such that $f(x) - \lambda \geq 0$ over \mathcal{C} , relating polynomial optimization and nonnegativity certificates. Typical relaxations are Sums of Squares¹, leading to a convex optimization problem that can be solved via semidefinite programming, and Sums of Nonnegative Circuits². Such relaxations provide lower bounds on the global optimum of (1).

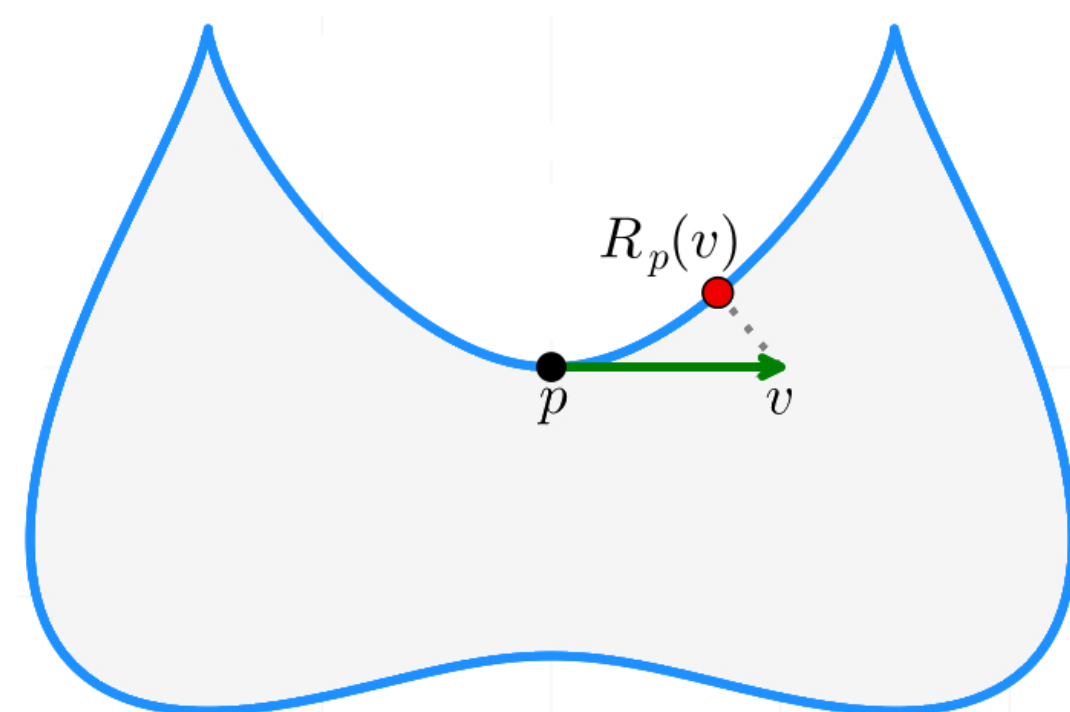
For finding upper bounds, currently either a highly specialized approach is employed³ or Newton's method is directly applied. This approach is not guaranteed to converge globally to the closest minimum.

3. Euclidean Distance Retraction

The closest point problem:

For a smooth $\mathcal{C} = g^{-1}(0) \cup \{x : h_j(x) \geq 0\}$ in \mathbb{R}^n and a fixed point $u \in \mathbb{R}^n$, the closest point problem can be expressed as

$$\begin{aligned} \min_x & \frac{1}{2} \|x - u\|^2 \\ \text{s.t. } & g_i(x) = 0 \text{ and } h_j(x) \geq 0 \\ & \text{with } i \in \mathcal{E} \text{ and } j \in \mathcal{I}. \end{aligned}$$



In terms of *Karush-Kuhn-Tucker conditions*⁵, this problem can be reformulated as

$$\mathcal{L}(x, \lambda, \mu; u) = \frac{1}{2} \sum_{i=1}^n (x_i - u_i)^2 - \lambda^T g(x) - \mu^T h(x).$$

In $p \in \mathcal{C}$, the set of *active indices* is $\mathcal{A}(p) = \{i : i \in \mathcal{E} \text{ or } h_i(p) = 0\}$. From now on, assume that the set of gradients of active constraints in p is linearly independent (LICQ). The first-order optimality criterion then becomes

$$\begin{aligned} \nabla_x \mathcal{L}(x, \lambda, \mu; u) &= 0 \\ g_i(x) &= 0 \text{ and } h_j(x) \geq 0, \\ \mu_j &\geq 0 \text{ and } \mu_j \cdot h_j(x) = 0. \end{aligned}$$

In particular, the Lagrange multipliers corresponding to the inactive inequality constraints are zero. Hence, we can "track" the known solution $x = p$ along a path $u : [0, 1] \rightarrow \mathbb{R}^n$ from $u_1 = p$ to $u_0 = p + v$ via the straight-line homotopy

$$H(x, \lambda, \mu; t) = \nabla \mathcal{L}(x, \lambda, \mu; (1-t)u_0 + tu_1)$$

using a predictor-correct scheme known as *homotopy continuation*⁶. It uses an iterative combination of Euler's and Newton's method; whenever $h_j(x) < 0$, we add it to the equality constraints to correct the point to the boundary. Otherwise, h_j does not contribute to this step. Certified curve-tracking methods are applicable⁷.

4. Optimality Condition over Semialgebraic Sets

Since the LICQ holds, the *tangent cone* in p is equal to⁵

$$\hat{T}_p \mathcal{C} = \{w : w^T \nabla g_i(p) = 0 \text{ for } i \in \mathcal{E} \text{ or } w^T \nabla h_i(p) \geq 0 \text{ for } j \in \mathcal{A}(p) \cap \mathcal{I}\}.$$

Theorem⁵: Assume for some $p \in \mathcal{C}$ there is a Lagrange multiplier λ, μ such that the KKT conditions are satisfied. If additionally,

$$w^T \nabla_{xx}^2 \mathcal{L}(p, \lambda, \mu) w > 0 \text{ for all nonzero } w \in \hat{T}_p \mathcal{C}$$

with $w^T \nabla h_j(p) = 0$ for all j with $\mu_j > 0$, then p is a strict local minimum.

2. Riemannian Geometry

Definition: Let $\mathcal{S} \hookrightarrow \mathcal{M}$ be an embedded Riemannian manifold and $x \in \mathcal{M}$. A *retraction* at x is a smooth map $R_x : T_x \mathcal{M} \rightarrow \mathcal{M}$ such that each curve $c(t) = R_x(tv)$ satisfies

$$c(0) = x \text{ and } \dot{c}(0) = v.$$

If additionally $\ddot{c}(0) \in N_x(\mathcal{S}, \mathcal{M})$, it is called *2nd-order*.

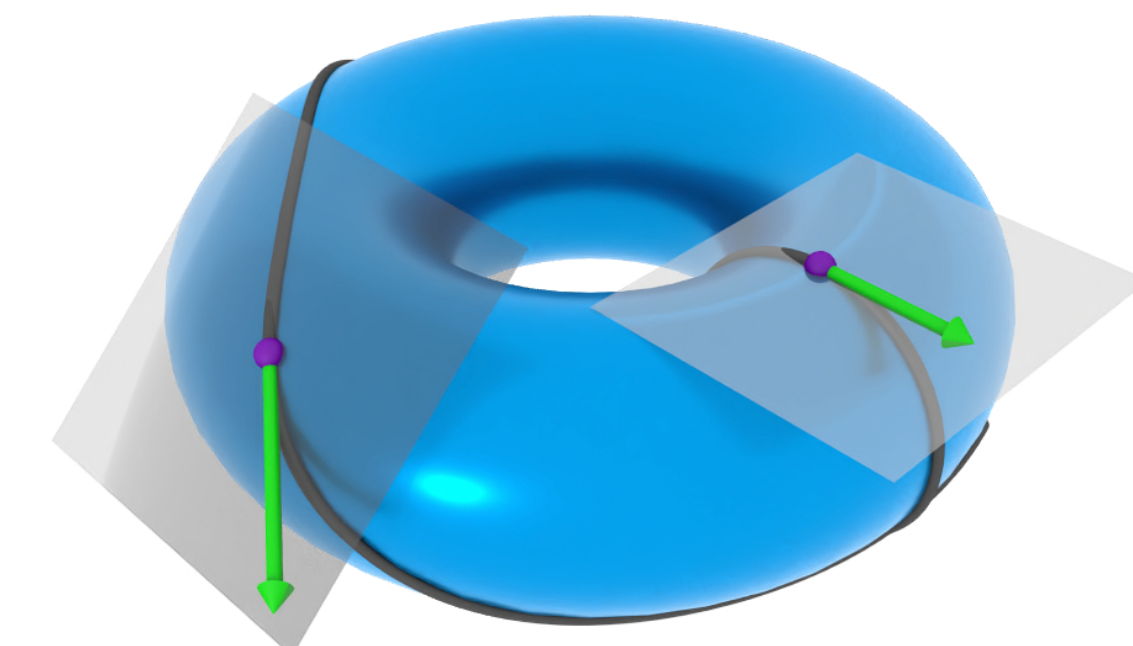


Figure: Two second-order retractions on the torus.

Theorem⁴: Let $\mathcal{M} \subset \mathbb{R}^n$ be a smooth manifold. For any point $p \in \mathcal{M}$, define the relation $R_p \subset T_p \mathcal{M} \times \mathcal{M}$ by

$$R_p = \{(v, u) \in \mathbb{R}^n \times \mathbb{R}^n : u \in \operatorname{argmin}_{y \in \mathcal{M}} \|p + v - y\|\}.$$

There exists a neighborhood U of 0 in $T_p \mathcal{M}$ such that R_p defines a local, second-order retraction. I.e., the curve $t \mapsto R_p(tv)$ matches the geodesic at p corresponding to v up to second-order.

5. Riemannian Gradient Descent

For a Riemannian manifold \mathcal{M} , consider the constrained optimization problem

$$\min_{x \in \mathcal{M}} f(x)$$

with a smooth $f : \mathcal{M} \rightarrow \mathbb{R}$. Similar to unconstrained optimization, local methods such as gradient descent can be employed to solve this problem⁸: Given a descent direction $v_k \in \hat{T}_{x_k} \mathcal{M}$ satisfying the conditions from Zoutendijk's theorem⁹ and a step size α_k satisfying the Wolfe conditions⁹, the retraction is applied repeatedly until f 's Riemannian gradient is smaller than a given tolerance $\tau > 0$:

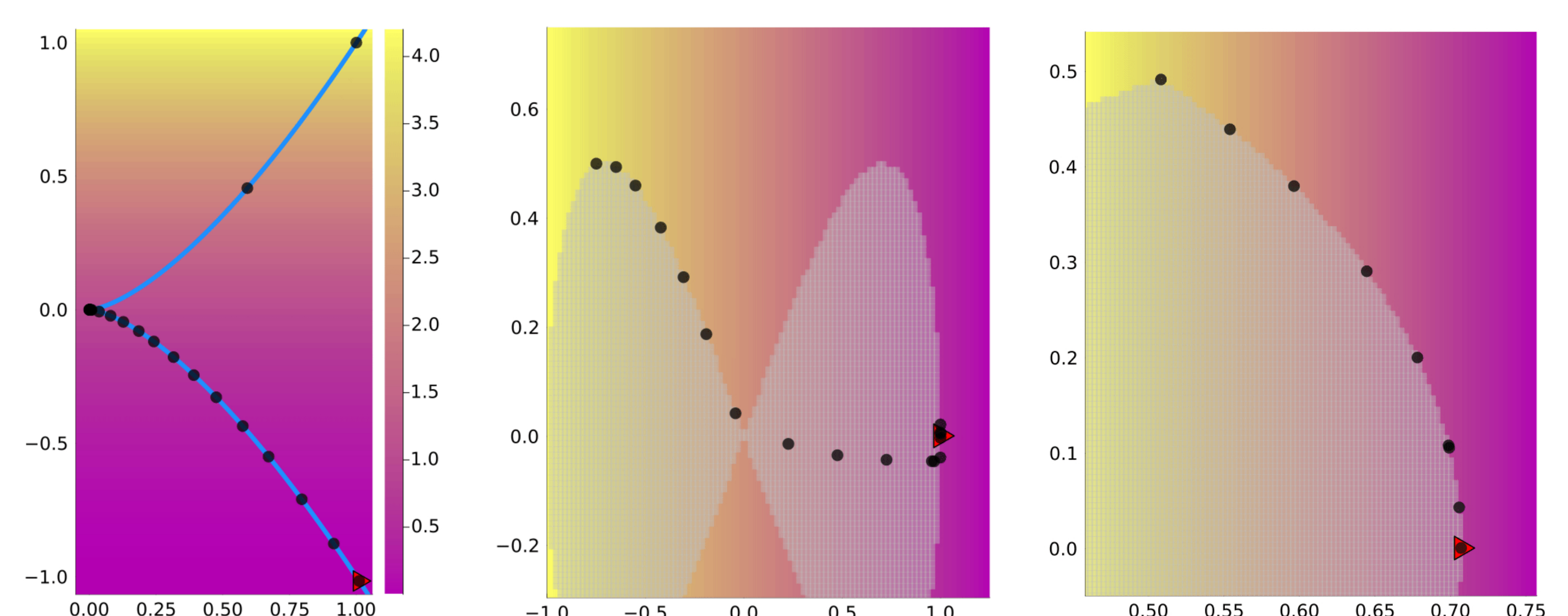
$$x_{k+1} = R_{x_k}(\alpha_k v_k) \text{ until } \|\operatorname{grad} f(x_k)\| < \tau.$$

Here, the Riemannian gradient $\operatorname{grad} f(x)$ is given by the projection of $\nabla f(x)$ onto the tangent cone $\hat{T}_x \mathcal{M}$. In the case of semialgebraic sets \mathcal{C} , this is the projection onto the linear subspace generated by the equality and the violated inequality constraints $v^T \nabla h_j(x) < 0$ with $j \in \mathcal{A}(x) \cap \mathcal{I}$ for a descent direction $v \in \hat{T}_x \mathcal{C}$.

6. Singularities

In singularities, the dimension of the tangent space may be larger than expected. Hence, there may only be a subset of tangent directions with measure zero that linearizes the local behavior of the constraint set. Given a semialgebraic set \mathcal{C} , the set of singular points has a positive codimension and the regular loci of the irreducible components form smooth manifolds¹⁰. By taking random and small tangent steps weighted by the Riemannian gradient in the singularity, we are able to reliably escape singularities. The optimization routine proceeds if a lower objective function value is identified; otherwise, the singularity is considered optimal.

7. Examples



[†]This project is joint work with Timo de Wolff.

¹ D. Hilbert: Über die Darstellung definiter Formen als Summe von Formenquadraten. Math. Ann. **32.3** (1888).

² S. Ilmanen, T. de Wolff: Amoebas, Nonnegative Polynomials and SOS Supported on Circuits. R. Math. Sc. **3.1** (2016).

³ J. B. Lasserre: A New Look at Nonnegativity on Closed Sets and Polynomial Optimization. SIAM J. Opt. **21.3** (2011).

⁴ P.-A. Absil and J. Malick: Projection-like Retractions on Matrix Manifolds. SIAM J. Opt. **22.1** (2012).

⁵ J. Nocedal, S. Wright: Numerical Optimization. 2nd ed. Springer Series in Op. R. and Fin. Eng. (2006).

⁶ A.J. Sommese, J. Verschelde, C.W. Wampler: Introduction to Numerical Algebraic Geometry. Springer (2005).

⁷ M. Burr, M. Byrd, K. Lee: Certified algebraic curve projections by path tracking. ISSAC (2025).

⁸ N. Boumal. An introduction to optimization on smooth manifolds. Cambridge University Press (2023).

⁹ W. Ring and B. Wirth. Optimization methods on Riemannian manifolds [...]. SIAM J. Opt. **22.2** (2012).

¹⁰ J. Milnor: Singular Points of Complex Hypersurfaces. Princeton University Press (1968).